4. SANNIKOV V.F. and CHERKESOV L.V., Development of spatial internal waves in a flow of stratified fluid. In: Internal and Surface Waves. Sevastopol, Izd. Mor. gidrofizich. in-ta Akad. Nauk SSSR, 1981.
5. GRAY E.P., HART R.W. and FARELL R.A., The structure of the internal wave Mach front generated by a point source moving in a stratified fluid. Phys. Fluids, 26, 10, 1983.
6. BOROVIKOV V.A., VLADIMIROV YU.V. and KEL'BERT M.YA., The internal gravitational wave field excited by localized sources. Izv. Akad. Nauk SSSR, Fizika atmosfery i okeana, 20, 6, 1984.
7. SANNIKOV V.F., The near field of stationary waves generated by a local perturbation source in a flow of stratified fluid. In: Theoretical Studies of Wave Processes in Ocean, Sevastopol, Izd, Mor. gidrofizich. in-ta, Akad. Nauk SSSR, 1983.
8. ABRAMOVITZ M. and STEGUN I.A., (Ed.). Handbook of Mathematical Functions. N.Y., Dover, 1956.
9. FEDORYUK M.V., The Saddle Point Method. Moscow, Nauka, 1977.
10. SANNIKOV V.F., Internal stationary waves generated by a local perturbation source within a flow, In: Modelling of Internal and Surface Waves. Sevastopol, Izd. Mor. gidrofizich. in-ta, Akad. Nauk SSSR, 1984.

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DIFFUSION ANALOGUE OF A COMBUSTION WAVE IN A SYSTEM WITH DISCRETE SOURCES*

G.G. CHERNAYA

The problem of selfsustaining concentration waves in discrete quasione-dimensional system with diffusion and threshold activation of the sources is considered. A number of applications of such models for describing spontaneous contraction waves observed in the course of experiments involving single muscle cells is discussed.

For many bological objects the passive transport of matter (caused by a difference in concentration) across some barriers such as biological membranes, depends in a complex manner on the absolute values of the concentrations on both sides of the barrier. Normally this is caused by the fact that transported material takes part in the chemical reactions which radically alter the effective permeability of the barrier.

One of the most interesting processes of this type is the release of Ca^{2+} ions from the inner cavities (terminal cisterns) of the cardiac musle cell directly into the contractile apparatus, the release occurring when the concentration of these ions outside the cavities reaches some threshold value. It is also necessary, in order for the release to occur, that this external concentration should approach its threshold value from below and at a sufficently rapid rate /1, 2/. The membrane which confines the intracellular cisterns, distributed within the cells in an orderly manner and separated from each other by distances of order at least equal to the size of the cisterns themselves, is regarded as the barrier.

The successive release of the ions from the cisterns can occur either as the result of diffusion of the released calcium, or with the help of electric control signals /3/ propagating rapidly along the cell. The signal can appear, in principle, as a result of large changes in ion concentrations occurring after calcium has been released from the cisterns. The release of calcium from the sequentially distributed cisterns results in the formation of a wave of increased concentration of calcium propagating along the cell, followed by a mechanical concentration wave which was observed experimentally in /4/ (the Ca^{2+} ions locally trigger the performance of contractile structures of the cell).

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The mechanisms of propagation of the wave of increased calcium concentration described above show analogies with slow combustion and detonation phenomena in continua /5, 6/. However, direct application of the corresponding mathematical models is hampered by the fact that the finite distances between the cisterns, which are much too large to accommodate the equations with continuously distributed parameters, must be taken into account. At the same time, these distances are insufficiently large to restrict the investigation to the interaction of two adjacent cisterns only, as was done in certain mathematically similar problems of the theory of nervous impulse /3/.

The present paper deals with the basic problems of concentration waves in a discrete, quasione-dimensional system on a half-line and on a segment, using the approximation in which the cisterns are replaced by regularly distributed point sources. Special attention is given, when discussing the solutions, to the laws governing the propagation of concentration waves.

1. General formulation of the problems. The diffusion in a binary non-deformable continuum containing continuously distributed sinks and discrete sources,is described by the following equation for the concentration $C = C(t, \mathbf{r})$:

$$\Psi \frac{\partial C}{\partial t} = \operatorname{div}(\mathbf{D} \operatorname{grad} C) - k^{(i)}C + \sum I_s^{(i)}\delta(\mathbf{r}-\mathbf{r}_s) \tag{1.1}$$

Here $\mathbf{r}_s$ are the coordinates of the s-th source of intensity $I_s^{(i)}(t)$, $\delta(\mathbf{r}-\mathbf{r}_s)$ is the three-dimensional delta function, $\mathbf{D}$ is the tensor of diffusion coefficients, $k^{(i)}$ is the capacity of the sinks and Ψ is a coefficient characterizing the instantaneous reversible binding of the diffusing material caused by the chemical reactions. In order to be able to obtain an analytic solution of the problem, we will henceforth assume that $\mathbf{D}$, Ψ and $k^{(i)}$ are constant. The functions $I_s^{(i)}(t)$ are, in general, non-linear functionals of C. Below we postulate, for simplicity, the threshold character of activation of the sources, i.e.

$$I_r^{(i)}(t) = \sum_{s=1} I_{rs}^{(i)}(t-t_{rs}) H(t-t_{rs}) \tag{1.2}$$

$$C(t_{rs}, \mathbf{r}_r) = C_\varepsilon, \quad \left.\frac{\partial C(t, \mathbf{r}_s)}{\partial t}\right|_{t=t_{rs}} \geqslant v \geqslant 0 \tag{1.3}$$

Here C_ε and v are the threshold values of the variables, H is the Heaviside function, t_{rs} is the instant of s-th activation of the r-th source (in the abbreviated notation $t_{r1} = r_r$). We further restirct the analysis to the case when $I_{rs}^{(i)} = I^{(i)}$ for all r, s and $v = 0$. Eq.(1.1) is solved for theregion of prescribed configuration under the initial condition $C = C_0 = \text{const}$ and prescribed flux of the material across the boundary.

Let the diffusion occur in a cylindrical region of radius R and length L, with $L \gg R$. It will be appropriate to use the approximate quasione-dimensional formulation of the problem in terms of the quantities averaged over the cross-section of the cylinder. Let the flux across the side surface of the cylinder be linearly related to the concentration, and let the sources be distributed in the planes $x_r = r\delta$ so as to form n equally oriented rows along lines parallel to the x axis. Then, passing in (1.1) to cylindrical coordinates and integrating over the cylinder cross-section we obtain, taking into account (1.2).

$$\Psi \frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} - k(C - C_0) + \sum_{r,s=1} I(t-t_{rs}) H(t-t_{rs}) \delta(x-x_r) \tag{1.4}$$

In (1.4) and from now on, C will denote the concentration averaged over the cross-section. $I = I^{(i)}n/\pi R^2$, D is the coefficient of longitudinal diffusion and $k \neq k^{(i)}$ in the effective capacity of the sinks.

The initial value of the concentration C_0 is assumed to be equal to the stationary concentration, and condition (1.3) takes the form

$$C(t_{rs}, x_r) = C_\varepsilon, \quad \left.\frac{\partial C(t, x_r)}{\partial t}\right|_{t=t_{rs}} \geqslant 0$$

At the initial instant $C(0, x) = C_0 = \text{const}$, and $C_0 < C_\varepsilon$. The flow $I_0 = -DC_x(t, 0)$ is specified at the boundary $x = 0$, and the boundary $x = L$ is assumed to be impermeable. The function $I(t)$ is assumed integrable over the whole t semi-axis, and the corresponding integral characterizing the total amount of material merging from a single source is denoted by I^*. No special assumptions are made concerning the integrability of $I_0(t)$.

2. Solution of the problem on a half-line. Let $L \gg \delta$, i.e. let the region in question contain sufficiently many sources so that when the values of x are not excessively large, then it is natural to solve Eq.(1.1) on the half-line $x \geqslant 0$, requiring that the solution have a limit as $x \to \infty$. Let us introduce the dimensionless variable and parameters

$$t^\circ=\frac{t}{t_*},\quad x^\circ=\frac{x}{x_*},\quad C^\circ=\frac{C-C_0}{C_*} \tag{2.1}$$
$$I^\circ=\frac{I}{I_*},$$
$$I_0^\circ=\frac{I_0}{I_{0*}},\quad k^\circ=\frac{kt_*}{\Psi}$$

We choose the following scales: for x we choose the distance between the sources $x_* = \delta$, for t the diffusion time $t_* = \Psi\delta^2/(4D)$, for C the quantity $C_* = 4I^*t_*/(\Psi\delta) = I^*\delta/D$ and for I and I_0 the quantity $I_* = I^*/t_*$. We omit the superscript on the dimensionless variables. We have the following expressions in dimensionless variables:

$$\frac{\partial C}{\partial t}=\frac{1}{4}\frac{\partial^2 C}{\partial x^2}-kC+\frac{1}{4}\sum_{r,\,s=1} I(t-t_{rs})H(t-t_{rs})\delta(x-x_r) \tag{2.2}$$

$$C(t_{rs},x_r)=C_\varepsilon,\quad \left.\frac{\partial C(t,x_r)}{\partial t}\right|_{t=t_{rs}}\geqslant 0,\quad C(0,x)=0 \tag{2.3}$$

$$I_0(t)=-\left.\frac{\partial C}{\partial x}\right|_{x=0},\quad C<\infty \quad \text{as} \quad x\to\infty \tag{2.4}$$

Since the problem is linear, we have

$$C=C_0(t,x)+\sum_r C_r(t,x)$$

where C_0 and C_r are solutions of the problem for the half-line without any sources, and for a half-line with a single source at $x = x_r$, respectively, but with zero derivative $\partial C/\partial x$ at the point $x = 0$.

When $x \neq x_r$, we can write in place of (2.2),

$$\frac{\partial C}{\partial t}=\frac{1}{4}\frac{\partial^2 C}{\partial x^2}-kC \tag{2.5}$$
$$\{C\}=0,\quad \left\{\frac{\partial C}{\partial x}\right\}=-I(t-t_{rs})H(t-t_{rs}) \quad \text{when } x=x_r$$

Here $\{\ \}$ denotes a jump in the values of the function. Let us apply the Laplace transform to (2.5) and to the boundary conditions, taking into account the initial condition and denoting the transforms of C and I by $C^\times$, $I^\times$ with the corresponding indices. Then we have for $C^\times$, $C_0^\times$, $C_r^\times$

$$pC^\times=\frac{1}{4}(C^\times)''-kC^\times;\quad (C_0^\times)',\ (C_r^\times)'\to 0 \quad \text{as} \quad x\to\infty$$
$$(C_0^\times)'=-I_0^\times,\quad (C_r^\times)'=0 \quad \text{as} \quad x=0;$$
$$\{C_r^\times\}=0,\ \{(C_r^\times)'\}=-I^\times e^{-pt_{rs}} \quad \text{as } x=x_r$$

We will restrict ourselves (see Sect.4) to analysing the events taking place when the sources are activated only once, putting $t_{rs} = t_r$. Taking into account the contributions from all sources and carrying out the necessary reduction, we can write the transform at the point $x = x_s$ in the form

$$C^\times(p,x_s)=\frac{I_0^\times}{i\varkappa}\exp(-i\varkappa x_s)+\frac{I^\times}{i\varkappa}\Big\{\sum_{r=1}^{s-1}\cos\varkappa x_r\exp(-pt_r-i\varkappa x_s)+$$
$$\sum_{r=s}^{\infty}\cos\varkappa x_s\exp(-pt_r-i\varkappa x_r)\Big\};\quad \varkappa=2i\sqrt{p+k}$$

The factors accompanying $I_0^\times$ and $I^\times\exp(-pt_r)$ are, respectively, the transforms of the functions

$$G_0(t,x_s)=\frac{1}{\sqrt{\pi t}}\exp\left(-kt-\frac{x_s^2}{t}\right) \tag{2.6}$$
$$G_r(t,x_s)=\frac{1}{2\sqrt{\pi t}}\exp(-kt)\Big\{\exp\Big[-\frac{(x_s-x_r)^2}{t}\Big]+$$
$$\exp\Big[-\frac{(x_s+x_r)^2}{t}\Big]\Big\}$$

When $t < t_s$, we have, according to the convolution theorem,

$$C(t,x_s)=\int_0^t I_0(t-\tau)G_0(\tau,x_s)\,d\tau+\sum_{r=1}^{s-1}\int_0^t I(t-t_r-\tau)H(t-t_r-\tau)G_r(\tau,x_s)\,d\tau \tag{2.7}$$

We shall use this general formula in Sect.4 to find t_1 depending on the parameters of the problem, from the condition $C(t_1, x_1) = C_\varepsilon$. In addition, we shall use the following considerations to find the velocity of the source activation wave. Let x_s and the corresponding t_s be sufficiently large. Then the initial distrubiton C and the condition at $x = 0$ will have a vanishingly small effect on the solution. We see from (2.6) and (2.7) that the principal contribution to the function $C(t_s, x_s)$ is made by the terms of the series with $r = s - 1$, than by those with $r = s - 2$, etc. The second term within the curly brackets in (2.6) becomes vanishingly small as $s \to \infty$ compared with the first term. If we assume that the source activation wave exists as $x \to \infty$ and propagates with constant dimensionless velocity

$$c_\infty = \lim_{s\to\infty} (t_s - t_{s-1})^{-1} \tag{2.8}$$

then according to (2.7), the following formula will yield c_∞:

$$C_\varepsilon = \sum_{r=1}^{\infty} \int_0^{r/c_\infty} I\left(\frac{r}{c_\infty} - \tau\right) H\left(\frac{r}{c_\infty} - \tau\right) G(\tau, x_r)\, d\tau \tag{2.9}$$

$$G(\tau, x_r) = \frac{1}{2\sqrt{\pi t}} \exp\left[-\left(kt - \frac{x_r^2}{t}\right)\right]$$

Reductio ad absurdum can be used to show that the limit of (2.8), if it exists, is always finite and different from zero, i.e. when the sources are of equal strength, the wave can neither accelerate without limit, nor can its velocity tend asymptotically to zero.

3. Solution of the problem on a segment.

Using the same examples as in Sect.2 and retaining the same notation, we can construct the solution of (2.2) with conditions (2.3) and conditions at the boundary:

$$I_0(t) = -\frac{\partial C}{\partial x} \text{ when } x = 0, \quad \frac{\partial C}{\partial x} = 0 \text{ when } x = L$$

The following relation represents an analogue of (2.7):

$$C^\times(p, x_s) = \frac{I_0^\times}{\varkappa \operatorname{sh} \varkappa L} \operatorname{ch} \varkappa (L - x_s) + \frac{I^\times}{\varkappa \operatorname{sh} \varkappa L} \sum_{r=1}^{N} \exp(-pt_r) \times$$
$$[\operatorname{ch} \varkappa x_s \operatorname{ch} \varkappa (L - x_r) H(x_r - x_s) + \operatorname{ch} \varkappa x_r \operatorname{ch} \varkappa (L - x_s) H(x_s - x_r)]$$

Here N is the total number of sources equal, numerically, to the dimensionless length. The factors accompanying $I_0^\times$ and $I^\times \exp(-pt_r)$ are, respectively, the transforms of the functions

$$G_0 = \frac{1}{2N} \sum_{n=0}^{\infty} (-1)^n \cos\frac{\pi n}{N}(N - s) \exp(p_n t)$$

$$G_r = \frac{1}{2N} \sum_{n=0}^{\infty} (-1)^n \exp(p_n t) \left[H(r - s) \cos\frac{\pi n s}{N} \cos\frac{\pi n}{N}(N - r) + \right.$$
$$\left. H(s - r) \cos\frac{\pi n r}{N} \cos\frac{\pi n}{N}(N - s)\right]; \quad p_n = -k - \frac{\pi^2 n^2}{4N^2}$$

The formula (2.7) retains its form.

4. Discussion.

The parameters of the system in question which can influence the activation of the sources and the times $t_1, t_2, \ldots$, as well as the existence of the activation wave and its velocity, include C_ε, k, I, I_0. I in turn is expressed in terms of the parameters characterizing the amount of material I^* ejected from the source, the time of ejection, etc. The function I_0 can differ from it in the values of these parameters even when its functional form is exactly the same. The absolute (dimensional) values of $c_\infty, t_1, t_2, \ldots$ also depend on C_* and t_*, and $t_1, t_2, \ldots$ are proportional to t_*, with coefficients whose values are determined as the dimensionless times using, for example, the first formula of (2.3).

We will choose the following function (in dimensional variables) as the model of the dependence of I on t:

$$J_n(\alpha, t) = I^* \alpha^{n+1} t^n e^{-\alpha t} / n! \tag{4.1}$$

The function is characterized by three constants. The constant I^* represents the integral of J_n from 0 to ∞ (see (2.1)); α has the meaning of the characteristic time of variation of J_n, with α, n used to determine all time-related properties of J_n. When $n \to 0$ and $\alpha \to \infty$ we obtain $J_0(\infty, t) = J\delta(t)$. Apart from this case, we also discuss the functions $J_0(\alpha, t) = J\alpha e^{-\alpha t}$ and $J_1(\alpha, t) = J\alpha^2 t e^{-\alpha t}$. Using the dimensionless variables and replacing α by $\alpha^0 = \alpha t_*$, we find that the functions I in (2.2), (2.3) and (2.7) take one of the following three forms:

$$f_0(\infty) = \delta(t), \quad f_0(\alpha) \equiv \alpha e^{-\alpha t}, \quad f_1(\alpha) \equiv \alpha^2 t e^{-\alpha t} \tag{4.2}$$

In the numerical computations which follow, we shall use all three formulas of (4.2) and vary the parameters α and k. As regards the function I_0, we will assume that it is proportional to one of these functions with the coefficient J equal to the ratio of the strengths of the corresponding sources.

Using (4.2), we can compute the corresponding convolution integrals appearing in (2.7). They are expressed by the functions

$$F_0(\infty, k, \beta, t) = \frac{1}{\sqrt{\pi t}} \exp\left[-\left(kt + \frac{\beta^2}{t}\right)\right] \tag{4.3}$$

$$F_0(\alpha, k, \beta, t) = \frac{\alpha e^{-\alpha t}}{2a} \{f_0^+ + f_0^-\} \tag{4.4}$$

$$f_0^{\pm} = e^{\pm 2a\beta}\left[\operatorname{erf}\left(a\sqrt{t} \pm \frac{\beta}{\sqrt{t}}\right) \mp 1\right]$$

$$F_1(\alpha, k, \beta, t) = \frac{\alpha^2 e^{-\alpha t}}{2a^3}\left\{f_1^+ + f_1^- + \frac{2}{\sqrt{\pi}} a\sqrt{t}\exp\left(-a^2 t - \frac{\beta^2}{t}\right)\right\} \tag{4.5}$$

$$f_1^{\pm} = e^{\pm 2a\beta}\left(a^2 t \pm a\beta - \frac{1}{2}\right)\left[\operatorname{erf}\left(a\sqrt{t} \pm \frac{\beta}{\sqrt{t}}\right) \mp 1\right]$$

where $a = \sqrt{k - \alpha}$, is the error integral. We note that using the formulas given we can construct the solution for a more general case, when $I_0,\ I \sim f_n(\alpha) = \alpha^n t^n e^{-\alpha t}/n!$, using the equation

$$f_n(\alpha) = (-1)^n \frac{\alpha^{n+1}}{n!} \frac{\partial^n}{\partial \alpha^n} \frac{f_0(\alpha)}{\alpha}$$

The integrals appearing in (2.7) are obtained here with the help of differentiation with respect to a parameter and are expressed, in the end, for any n, in terms of the error integral and elementary functions.

When $t < t_1$, the concentration at the point x_1 is given by

$$C(t, x_1) = JF_0(\infty, k, 1, t), \quad I_0 = Jf_0(\infty) \tag{4.6}$$

$$C(t, x_1) = JF_0(\alpha, k, 1, t), \quad I_0 = Jf_0(\alpha) \tag{4.7}$$

$$C(t, x_1) = JF_1(\alpha, k, 1, t), \quad I_0 = Jf_1(\alpha) \tag{4.8}$$

For short t this yields

$$C(t, x_1) \approx \frac{\alpha^n J t^{2n-1/2}}{\sqrt{\pi}} e^{-1/t} \tag{4.9}$$

where $n = 0, 1, 2$ respectively.

The same result is obtained from the solution of the problem for a segment (Sect.3) when $x \ll L$. At small values of α $(\alpha \ll k)$ the parameter $a \approx \sqrt{k}$ is real. At fixed $t > 0$ the expression within the curly brackets in (4.4) or (4.5) tends to a finite limit as $\alpha \to 0$, therefore $C(t, x_1)$ decreases as α and α^2.

As $\alpha \to k$, the parameter a tends to zero and expressions (4.4) and (4.5) become indeterminate. We use standard methods to obtain, as $\alpha \to k$,

$$\begin{aligned} C(t, x_1) &= 2Je^{-\alpha t}[\operatorname{erf}(1/\sqrt{t}) - 1 + \pi^{-1}\sqrt{t}e^{-1/t}] + \\ &\quad O(a^2), \quad I_0 = Jf_0(\alpha) \\ C(t, x_1) &= {}^2/_3\, J\alpha^2 e^{-\alpha t}\{(2 + 3t)[\operatorname{erf}(1/\sqrt{t}) - 1] + \\ &\quad (2/\sqrt{\pi})\sqrt{t}(1 + t)e^{-1/t}\} + O(a^2), \quad I_0 = Jf_1(\alpha) \end{aligned} \tag{4.10}$$

When $\alpha > k$, the quantity a becomes purely imaginary $(a = i\gamma)$ and increases in modulo together with α. In order to investigate the behaviour of the quantities defined by formulas (4.4) and (4.5), we note tbat $\operatorname{erf}(\beta/\sqrt{t} + i\gamma\sqrt{t}) = -\overline{\operatorname{erf}}(i\gamma\sqrt{t} - \beta/\sqrt{t})$. From this we can conclude that the right-hand sides of (4.4) and (4.5) are real, and the expressions within the curly brackets are purely imaginary.

In the discussion that follows, we shall use the results of numerical computations carried out on a programmable calculator using the approximate formulas for $\operatorname{erf}(z)$ and the real and complex values of the argument /7/. Some of these data have already appeared in /4, 8/.

Figs.1-3 show, for the case $J = 1$, families of curves representing the dependence of C on t when $x = x_1$, with the parameters α, k, obtained from the numerical formulas (4.6)-(4.8). The graphs can be used at a fixed value of $C = C_s$ to find the time t_1 necessary to attain the threshold concentration C_ε at the point x_1.

Of the two branches of the curves shown in the graphs, the physically meaningful branch is the ascending one, corresponding to the change in the concentration C from the lowest values, to the maximum possible threshold value $C_{\varepsilon m}$. In the graphs $C_{\varepsilon m}$ coincides with the maxima of the curves. Raising the threshold value above $C_{\varepsilon m}$ implies, from the mathematical point of view, that the transcendental equations $C(t, x_1) = C_\varepsilon$ have no solutions, and from the physical point of view it implies that the amount of material passing across the cross-section $x = 0$ is insufficient to trigger the first source, and hence insufficient for the source activation wave to exist.

The graphs in Fig.1 correspond to formulas (4.6) and illustrate the influence of the parameter k. The general shape of the curves and the manner of their distribution are also preserved for the solutions (4.7) and (4.8) for fixed α (see the dashed curves in Fig.2 and 3); when k increases, t_1 also increases and $C_{\varepsilon m}$ decreases. The influence of the parameter α is illustrated by the solid lines in Fig.2 and 3 which were constructed for $k = 0.025$ using formulas (4.7) and (4.8) respectively.

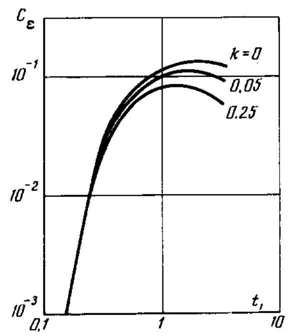

Fig.1

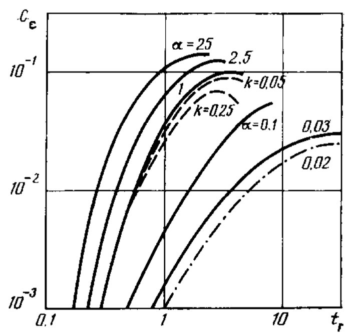

Fig.2

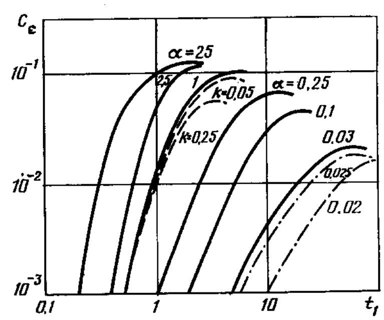

Fig.3

We see from the graphs that the time t_1 taken for a single particular value of C_ε increases as α decreases, and the value of $C_{\varepsilon m}$ decreases. This can be given a simple physical interpretation. When the material is brought in more slowly, the time t_1 necessary for the value of C_ε to be reached at the point x_1 increases, and $C_{\varepsilon m}$ decreases due to the occurrence of "elution". Changing the form of the function f (see (4.2)) alters the values of t_1 and $C_{\varepsilon m}$ over a narrow range.

The graphs show that for one and the same set of parameters the time t_1 cannot exceed some value t_{1m} representing the abscissa of the maximum of the curve. The latter means that when conditions necessary for activation of the source are established in the system, this will occur within a period shorter than the time needed to attain the maximum threshold value $C_{\varepsilon m}$ possible in the situation.

Figs.2 and 3 show also, for comparison, the results of computations using the asymptotic formulas (4.10) when $\alpha \to k$ (the dot-dash lines). We see from the graphs that the asymptotic form can be regarded as acceptable. The asymptotic form for the short times $t \leqslant 1/k$ (see (4.9)) is applicable, over a wide range of t, only for the case of $I_0 \sim \delta(t)$.

We obtain, in accordance with (2.9), the following formulas for the dimensionless velocity c_∞, introduced using (2.8), analogous to (4.6)-(4.8):

$$C_\varepsilon = \sum_{m=1}^{\infty} F_0(\infty, k, m/c_\infty, mc_\infty), \quad I_0 = f_0(\infty) \tag{4.11}$$

$$C_\varepsilon = \sum_{m=1}^{\infty} F_0(\alpha, k, m/c_\infty, mc_\infty), \quad I_0 = f_0(\alpha)$$

$$C_\varepsilon = \sum_{m=1}^{\infty} F_1(\alpha, k, m/c_\infty, mc_\infty), \quad I_0 = f_1(\alpha)$$

Fig.4 shows a family of curves representing the dependence of C_ε and c_∞, obtained from the second formula of (4.11) for $k = 0.025$. The curves for other values of k and other forms of f differ little from the curves shown.

The graphs of $C_\varepsilon(c_\infty)$ resemble, by virtue of (2.8) the mirror images of the graphs of $C_\varepsilon(t_1)$ shown in Fig.1-3. In the present case only the descending branch of each curve has a physical meaning, and the maximum on the curve corresponds to the largest possible threshold value of the concentration allowing for the existence of the wave. If the conditions created in the system are suitable for the existence of the source activation wave, then the velocity of the wave propagation cannot fall below a certain value.

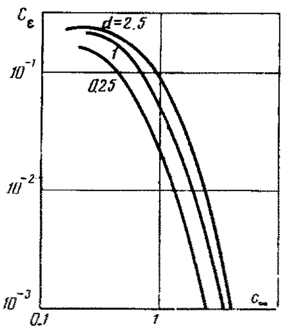

Fig.4

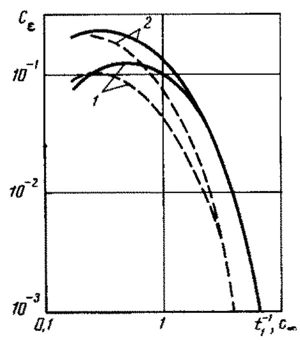

Fig.5

The curves shown in Fig.5 are of special interest in our discussion. They correspond to the graphs constructed using the same values of $k = 0.025$ and $\alpha = 1$, for the relations $C_\varepsilon(1/t_1)$ (curves *1*) and $C_\varepsilon(c_\infty)$ (curves *2*), with $J = 1/_2$, for $I \sim \delta(t)$ (solid lines) and $I \sim e^{-\alpha t}$ (dashed lines). We see that in both cases the "rate" $1/t_1$ at which any prescribed threshold value C_ε is attained at the position of the first source, is less than c_∞. This means that in the present case the wave accelerates as it moves. Fig.6a shows schematically, by dashed lines, the curves corresponding to the dependence of C_ε on $1/(t_2 - t_1)$, etc. They fill all space between the lines $C_\varepsilon(1/t_1)$ and $C_\varepsilon(c_\infty)$. Both solid lines in the figure attain their maximum values. For the first curve the maximum is equal to $C_{\varepsilon 1}$ and for the second curve it is $C_{\varepsilon m}$, with $C_{\varepsilon 1} < C_{\varepsilon m}$. If the given threshold value of the concentration $C_\varepsilon < C_{\varepsilon 1}$, conditions arise which are suitable for the activation of the first and all subsequent sources, i.e., the wave will exist, while if $C_{\varepsilon 1} < C_\varepsilon < C_{\varepsilon m}$, the wave cannot appear.

Taking into account the rapid convergence of series (2.9) and comparing its principal term $(r = 1)$ with the solution (2.7) at $t < t_1$ (first term), we can see that when $I_0 = JI$, then the values of C_ε computed from these formulas and far removed from $C_{\varepsilon m}$, will differ by a factor of approximately $2J$, for the numerically equal t_1 and $1/c_\infty$. It is this aspect that governed the choice of J made above. If we take $J < 1/_2$, then the relative position of the curves $C_\varepsilon(1/t_1)$ and $C_\varepsilon(c_\infty)$ will not be changed and the previous arguments will remain in force. If on the other hand we begin to increase the value of the parameter J (the coefficient of proportionality between I_0 and I) in the interval from $1/_2$ to 1, then the relative position of

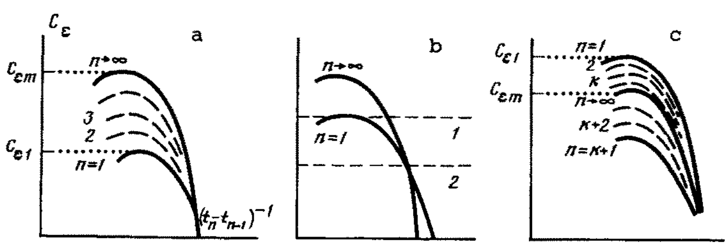

Fig.6

the curves will change and a point of their intersection will appear. Schematically, the graph will now resemble that of Fig.6b. It follows therefore that the range of values of C_ε will be split into two regions: in region *1* the wave will be accelerated, and in region *2* it will be retarded. Fig.6c shows schematically that when $J > 1$, several lines corresponding to the dependence of C_ε on $1/t_1, 1/(t_2 - t_1), \ldots$ will be situated above and to the right of all other lines, and will appear in the order reverse to the initial order. If we now choose C_ε so that $C_{\varepsilon m} < C_\varepsilon < C_{\varepsilon 1}$, then a situation will arise in which only the few first sources can be activated, i.e. a wave, having appeared and spread over some finite distance, will decay rapidly. The higher the value of $C_\varepsilon < C_{\varepsilon 1}$ chosen, the shorter the distance over which the wave will spread.

We note that the summation in (2.6), (2.9) and (4.11) reflects the contributions of the preceding sources towards the concentration field near the source under consideration. When $C_\varepsilon \ll C_{\varepsilon m}$, the number of terms of the series essential for the practical applications is small, but it grows as C_ε approaches $C_{\varepsilon m}$. The convergence of the series becomes slower, as we can see from, for example (4.3) and (4.11), as the wave velocity c_∞ decreases.

If we introduce varying values of α, and even more so functionally different emission laws for the first and for the following sources, then by suitable selection of the numerical coefficients we can obtain even more complicated patterns of relative distribution of the curves $C_\varepsilon(c_\infty)$ and $C_\varepsilon(1/t_1)$, $C_\varepsilon(1/(t_2-t_1))$, ... In particular, when $I_0 = If_0(\alpha_0)$, $I = f_0(\alpha)$, and $I \sim 1$, $\alpha_0 \ll \alpha$, then, provided that the given threshold value C_ε is not high, the time t_1 may be found to exceed c_∞^{-1} by 1-2 orders of magnitude. If we accept the existence of a mechanism which returns the system to its initial state after the wave has passed and then activates the source I_0 once again, then waves will be generated repeatedly and the period will be large, of the order $\geqslant t_1$. Such a mechanism can be described, in the simplest case, by an equation of the type $\partial I_0/\partial t = \theta(I_0, C(t, 0))$.

Using the models with threshold non-linearity, whose examples were given in Sect.1-3, we can study not only the situation with peridic excitation mentioned above, but also many other cases such as e.g. the propagation of a wave in a system which has a barrier offering high resistance to diffusion, which cannot let a wave through until the threshold value becomes sufficiently low. This also refers to the case when the waves are repeated with increased acceleration and frequency caused by the fact that each passage of a wave leads to an increase in the noise level, i.e. to a lowering of the threshold. Certain additional possibilities of using threshold-type models were discussed in /8/.

A preliminary analysis /4, 8/ has shown that mathematical models resembling those discussed above can be used to study spontaneous contraction waves in the cardiac muscle cells described at the beginning of this paper. The waves can be quite complicated at times, for example, situations can be observed when a wave generated at one end of the cell disappears after passing through a small part of its length. The experiments did not detect any acceleration in the wave as it moved along the cell. Not once has a contraction wave been detected whose velocity fell below a specified value (50 microns/sec). The solutions given above offer a possible explanation for this as well as for other special feature of the spontaneous waves.

A particular situation mentioned in the introduction, when a rapid electric signal appears (somewhat analogous to the process whereby slow combustion is transformed into a detonation), needs an essentially different model, namely, in addition to the relations connecting I with time, we need a relation connecting I with a certain parameter $\varphi(t, x)$ (cell membrane potential) which obeys a special parabolic equation with non-linear sources, the latter in turn depending on φ, C.

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REFERENCES

1. FABIATO A., Calcium-induced release of calcium from the cardiac sarcoplasmic reticulum. Amer. J. Physiol., 245, 1, 1983.
2. FABIATO A., Time and calcium dependence of activation and inactivation of calcium-induced release of calcium from the sarcoplasmic reticulum of a skinned canine cardiac Purkinje cell. J. Gen Physiol., 85, 2, 1985.
3. MARKIN V.S., PASTUSHENKO V.F. and CHIZMADZHEV YU.A., Theory of Excitable Media. Moscow, Nauka, 1981.
4. CHARNAYA G.G. and REGIRER S.A., On the mechanism of spontaneous contraction waves in isolated cardiomyocytes. In the book: Medicinal Biomechanics. 1, Riga, 1986.
5. SEDOV L.I., Mechanics of Continua. 1, Moscow, Nauka, 1983.
6. CHERNYI G.G., Exotheric waves in continua. In the book: Collected Problems of Modern Mechanics. Moscow, Izd-vo MGU, 1982.
7. Handbook of mathematical functions/Ed. by Abramovitz M., Stegun I.A.N.Y.: Dover Publ., 1956.
8. REGIRER S.A., TSATURYAN A.K. and CHERNAYA G.G., Mathematical model of the propagation of an activation wave in an isolated cardiomyocyte. Biofizika, 31, 4, 1986.

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